EXAMINING OPPORTUITES TO LEARN DEFINITE INTEGRALS IN WIDELY USED CALCULUS TEXTBOOKS

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This study explores opportunities to learn definite integrals in three widely used textbooks in the U.S. Definitions, worked examples, and exercise problems were coded using research – based cognitive resources in definite integrals. The results show that limited opportunities for students to explore multiplicative relationship between two quantities and adding small pieces to understand definite integrals.

Keywords: cognitive resources, integral, learning opportunities, textbooks

Introduction

In the United States, Calculus 1 is a core and beginning course for students heading into the disciplines of science, technology, engineering, and mathematics (STEM). Many experts in teaching calculus view an understanding of the integral as a necessary central concept or skill for genuine understanding of first-year calculus (Jones, 2013; Sofronas et al., 2011). While understanding of the integral is necessary, several existing studies demonstrate students' challenges as well as types of knowledge needed to understand the integral concept (Jones, 2013; Mahir, 2009; Sealey, 2014), illustrating that, in many cases, students are able to correctly integrate functions but do not know how to interpret it in context and tend not to use types of knowledge needed to understand integral conceptually. In this study, we examined how widely used calculus textbooks provide learning opportunities to students. These are our research questions that we attempted to answer.

- What units of knowledge about definite integral do widely used calculus textbooks provide to students?
- How do widely used calculus textbooks provide opportunities for students to be familiar with adding up pieces symbolic form to introduce the concepts of integral?

Related literature

How students learn integral concepts

According to experts in teaching calculus, there are three significant interpretations of definite integrals: 1) the integral as net change or accumulated total change, 2) the integral as area, and 3) techniques of integration (Ely, 2017; Sofronas et al., 2011). Previous studies show that students are able to procedurally calculate integrals but are not able to explain what they are doing and are not able to define the definite integral (Rasslan & Tall, 2002). What we can learn from these studies is students are very familiar with certain interpretations of the integral (area under a curve, anti – derivative, or techniques of integration), but it is challenging for them to understand why the definite integral represents area under a curve or in other contexts such as physics and engineering (Ely, 2017; Jones, 2015; Sealey, 2014). More recently, in addition to pointing out students' familiarity to certain interpretations of integrals, researchers have explored questions about types of knowledge students need and coordinate to expand their understanding

of definite integral (Larsen, Marrongelle, Bressoud, & Graham, 2016). This idea led to examination of small units of ideas that students need to activate to expand their understanding of definite integral (Jones, 2013) because it might be too simplistic to say that understanding mathematical topics requires a single idea (Harlow & Bianchini, 2020). With examination of small units of ideas, in addition to think about area under a curve, and anti – derivative, we can look at more fine-grained element of knowledge students need to coordinate to expand meaning of definite integrals. Sealey (2014) described four different layers − limit, summation, function, and product − in understanding definite integrals. Among these, students struggled most to understand the product layer that describes the product of two quantities, $f(x_i)$ and Δx (Sealey, 2014). Jones (2013) found students' various cognitive resources ("fine-grained" elements of knowledge in a person's cognition) in understanding integrals, claiming that students may demonstrate challenges in understanding integrals because productive cognitive resources were not activated. He observed a variety of cognitive resources activated by students in terms of symbolic forms, the perimeter and area symbolic form, the function matching symbolic form, and the *adding up pieces* symbolic form (Jones, 2013). Among these, Jones (2013) found that adding up pieces was most productive in understanding definite integral. Since recognizing the multiplicative relationship in adding up pieces is useful in applied contexts in physics and engineering (Ely, 2021; Jones, 2013; Sealey, 2014), with adding up pieces, students may possess units of knowledge that enable them to expand their understanding of definite integrals. Thus, being able to use adding up pieces along with the product layer may be a key units of knowledge students need to possess (Jones, 2015).

Exploring calculus students' learning opportunities

As Reeves, Carnoy, and Addy (2013) described, we can examine content coverage (list of topics and subtopics covered), content exposure (amount of time spent to instruction), and (c) content emphasis (which topics are selected for emphasis) to examine students' learning opportunities. When particular procedures, algorithms, and problems are presented in textbooks, teachers and students can have potential learning opportunities to become familiar with those procedures and problems. When teachers prepare and plan their lessons, they select (and modify) tasks and activities from textbooks in planning their lessons (Remillard & Heck, 2014). When selected tasks and activities are included in the lessons and enacted, what is included from textbooks and lessons transform into learning opportunities for students. Additionally, when some tasks are assigned as homework or when students read textbooks to study on their own, those can be transformed into learning opportunities for students. We think of calculus students' opportunities to become familiar with the concepts of definite integrals. If widely used textbooks give more attention to the perimeter and area symbolic form or the function matching symbolic form than adding up pieces (less coverage, exposure and emphasis) and calculus instructors use those ideas, students' understanding may be limited opportunities to those ideas that they learn from textbooks. Limited coverage in widely used textbooks may lead them to have units of knowledge allow them to interpret integrals as area under a curve or anti – derivative but be unable to interpret definite integrals in other contexts or not feel the need to learn and interpret integral in other contexts.

Methods

Data source

Three widely used Calculus 1 books, Calculus 7E 7th Edition (Stewart, 2010, abbreviated to SC), Calculus Single & Multivariable 7th Edition (Hughes-Hallett et al., 2017, abbreviated to HC), and Thomas' Calculus 13th Edition (Thomas et al., 2014, abbreviated to TC) were selected

for analysis (Bressoud, 2011), we analyzed Riemann sum and the first integral sections from each textbook. In total, 128 integral tasks (both worked examples and exercise problems) from Stewart, 156 tasks from Thomas, and 113 tasks from Hughes-Hallett et al. were analyzed.

Textbook analysis

We referred to several textbook content studies to shape our analytic framework (Charalambous et al., 2010; Hong, 2023; Hong, Choi, Runnalls, & Hwang, 2018; Son & Hu, 2016). First, how the three textbooks introduce definite integral concepts was examined. Second, we examined each worked example and exercise problem and coded each item (one item can be one worked example or one exercise problem). We were interested in finding problems and worked examples that explore adding up pieces and the product layer (multiplicative relationship) in addition to the familiar interpretations of area under a curve and anti-derivatives (or techniques of integrations).

Examples of coding

Table 1 are codes that we used and terms/mathematical expressions that we looked for. Examples of these codes are provided in this section.

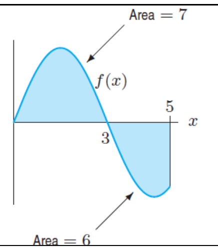
Table 1: Analytic Framework

Code	Terms/mathematical expression
Adding up pieces	Adding rectangles under a curve as the number of rectangles increases Multiplication between quantities relevant to definite integrals Meaning of each symbol
Area under a curve	Finding area under a curve without connecting it to adding rectangles
Anti-derivative (function matching)	Finding anti-derivative to evaluate integral
Matching mathematical symbols to integral	Matching each mathematical symbol in Riemann sum to integral without describing what each means
Evaluating Riemann sum or integral with finite number of rectangles	Evaluating Riemann sum or integral with several rectangles
Using integral properties	Verifying various properties of integrals

Table 2 shows examples of each code that we used for integral tasks from the three textbooks. These codes show types of knowledge that students will be exposed to when they use these textbooks.

Table 2: Coding examples

Item	Code
If your CAS can draw rectangles associated with Riemann sums, use it to draw rectangles associated with Riemann sums that converge to the integrals. Use n = 4, 10, 20, and 50 subintervals of equal length in each case. $\int_0^1 (x^2 + 1)dx = \frac{4}{3}$ (TC, p. 274)	Adding up pieces
What is the area between the graph of f(x) in Figure 5.35 and the x-axis, between x = 0 and x = 5? (HC, p. 287)	Area under a curve



Prove that $\int_a^b x\,dx = \frac{b^2-a^2}{2}$ (SC, p. 377)	Anti-derivative (function matching)
Express the limits in Exercises 1-8 as definite integrals. $\lim_{\lVert p\rVert\to 0}\sum_{k=1}^{n} c_k^2\,\Delta x_k$, where p is a partition of [0, 2] (TC, p. 274)	Matching mathematical symbols to integral
Estimate $\int_0^1 e^{x^{-2}}\,dx$ using n = 5 rectangles to form a left-hand sum (HC, p. 288).	Evaluating Riemann sum or integral using several rectangles
Use the properties of integrals to verify the inequality without evaluating the integrals. $\int_0^1 \sqrt{1+x^2}\,dx \le \int_0^1 \sqrt{1+x}\,dx$ (SC, p. 378).	Using integral properties

The first example shows that evaluating definite integral can be done with several rectangles. Students are asked to think about what will happen if more rectangles are added, which gives them ideas of adding up small pieces to get area. The second example is about area under a curve. This example does not ask students to relate the area to rectangles. It simply asks them to see the shaded regions represent the area under a curve. The third example can be answered if students know how to find an anti-derivative of x. Knowing how to find anti-derivative of x will be enough to answer this correctly. The fourth example asks students to match Riemann sum to definite integral, matching each symbol to the corresponding symbol of definite integral. Although they may see the product layer (how $f(c_k)$ are replaced $f(x)$ and how the subinterval widths Δx_k become the differential dx), it is possible that students just simply match each symbol in Riemann sum to each symbol in integral. The fifth example asks students to use five rectangles, a finite number, instead of conceptualizing many rectangles to estimate $\int_0^1 e^{x^{-2}}\,dx$. The last example asks students to verify an inequality using one of the properties of integral. This can be answered correctly once the graphs of two functions are drawn. This requires understanding of area, but this example may not lead students to think about infinitely many rectangles.

Reliability

We reviewed previous studies carefully to develop our codes (Jones, 2013; Sealey, 2014). We were able to develop a few codes as we examined textbooks. Once developed, two independent readers reviewed the textbooks several times and compared findings to ensure reliability of results. The initial agreement rate for task discussions was 95%. Discussion was done until there was 100% agreement.

Results

Introduction and definition of definite integral

Figure 1 shows the definition of definite integral from the three textbooks.

Stewart	Thomas	Hughes-Hallett

For now, *dx* the symbol has no meaning by itself; $\int_a^b \boldsymbol{f(x)dx}$ is all one symbol. The *dx* simply indicates that the independent variable is x. So Definition 2 says that the definite integral of an integrable function can be approximated to within any desired degree of accuracy by a Riemann sum. If the subinterval widths are $\boldsymbol{\Delta x_1, \Delta x_2 \cdots \Delta x_n}$, we have to ensure that all these widths approach 0 in the limiting process (p. 366- 368).	The sum symbol $\sum$ **is** replaced in the limit by the integral symbol $\int$, whose origin is in the letter "S" The function values $\boldsymbol{f(c_k)}$ are replaced by a continuous selection of function values $\boldsymbol{f(x)}$. The subinterval widths $\boldsymbol{\Delta x_k}$ become the differential *dx*. It is as if we are summing all products of the form $\boldsymbol{f(x)}$ • *dx* as *x* goes from *a* to *h*. (p. 263)	Now we take the limit of these sums as n goes to infinity. If f is continuous for a ≤ t ≤ b, the limits of the left- and right-hand sums exist and are equal. The *definite integral* is the limit of these sums. The "$\int$" notation comes from an old-fashioned "S," which stands for "sum" in the same way that $\sum$ **does.** The "dt" in the integral comes from the factor Δt (p. 282).

Figure 1: Introduction of definite integral

Prior to introducing definite integrals, all three textbooks had a section introducing Riemann sum and showed how $f(c_k) \cdot \Delta x_k$ would provide the area of each rectangle. Moreover, all three textbooks explored the distance traveled problems using the product layer, $f(c_k) \cdot \Delta x_k$. Additionally, SC and TC included several problems (similar to example 1 in Table 2) that asked students to use computer or graphing calculators to describe adding up pieces idea. We were very interested in exploring how three textbooks connect Riemann sum to integrand and differential in definite integrals. After introducing Riemann sum, all three textbooks describe the infinite process of dividing an interval to smaller pieces showing the hint of describing "adding up pieces." For example, SC describes Δx_n as widths approaching zero while HC describes *dt* in the integral comes from the factor Δt. However, only TC includes explanations of what $f(x) \cdot dx$ means and where *dx* came from (connecting $f(c_k) \cdot \Delta x_k$ and $f(x)$ • dx) while the other two textbooks' explanations are unclear about the product layer.

Worked example

Table 3 shows the distribution of worked examples by each code. In these worked examples, the adding up pieces symbolic form is not found. Figure 2 shows worked example from SC.

Table 3: Distribution of worked examples

Code	Stewart	Thomas	Hughes-Hallett
Adding up pieces	2	1	0
Area under a curve	2	3	3
Anti-derivative	0	0	0
Matching mathematical symbols to integral	2	0	0
Evaluating Riemann sum or integral with several rectangles	6	3	5

Using integral properties	3	2	0
Total	15	9	8

The worked example in Figure 2 shows how to set up an expression from an integral. It describes dividing an interval and adding those, but the product layer is not clearly described. It appears that students are asked to match each symbol of the integral to summation symbols. To complete this task, students refer to Theorem 4, which is about converting the definite integral to Riemann sum.

Set up an expression for $\int_1^3 e^x dx$ as a limit of sums.

Solution: Here we have $f(x) = e^x, a = 1, b = 2$ and $\Delta x = \frac{b-a}{n} = \frac{2}{n}$

So $x_0 = 1, x_1 = 1+\frac{2}{n}, x_2 = 1+\frac{4}{n}, x_3 = 1+\frac{6}{n},$ and $x_i = 1+\frac{2i}{n}$

From Theorem 4 we get $\int_1^3 e^x dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_{k,})\Delta x = \lim_{n\to\infty} \sum_{i=1}^{n} f(1+\frac{2i}{n})\frac{2}{n} =$ $\lim_{n\to\infty} \frac{2}{n}\sum_{i=1}^{n} e^{1+2i/n}$ (SC, p. 371)

Figure 2: A worked example from SC.

Exercise problems

Table 4: Percent distribution of exercise problems

Code	Stewart	Thomas	Hughes-Hallett
Adding up pieces	12 (11%)	18 (12.2%)	0
Area under a curve	14 (12.3%)	41 (27.9%)	35 (33.3%)
Anti-derivative	3(32.6%)	0	0
Matching mathematical symbols to integral	23 (20.3%)	9 (6.1%)	0
Evaluating Riemann sum or integral with several rectangles	46 (40.6%)	22 (14.9%)	70 (66.7%)
Using integral properties	15 (13.2%)	57 (38.8%)	0
Total	113	147	105

Table 4 shows the distribution of exercise problems from the three textbooks. Both SC and TC includes several problems similar to Figure 3. In the process of solving the problem in Figure 3, students need to think about the number of rectangles (adding up pieces). Thus, they can see that adding infinitely many rectangles will give the area under a curve. Although students will see the idea of adding up pieces, it is not clear if they are able to see the product layer. They may see it when they create rectangles, but it may be useful to ask students to describe the area of each rectangle in Figure 3, how each area can be expressed as a product of two quantities, and how the product is related to integrand and differential.

Exercises 95-98, use a CAS to perform 1he following steps:

Plot the functions over the given interval

Partition the interval into n = 100, 200, and 1000 subintervals of equal length, and evaluate the function at the midpoint of each subinterval.

Compute the average value of the function values generated in part (b).

Solve the equation f(x) = (average value) for *x* using the average value calculated in part (c) for the n = 1000 partitioning.

$\boldsymbol{f(x) = \sin x}$ on $\boldsymbol{[0, \pi]}$ (TC, p. 274)

Figure 3: Adding up pieces problem from TC

For SC, the most frequent task was evaluating Riemann sum with several rectangles.

Use the Midpoint Rule with the given value of to approximate the integral. Round the answer to four decimal places. $\int_2^{10} \sqrt{x^3+1}\,dx, \quad n = 4$ (SC, p. 377)

Figure 4: Exercise problem from SC

For TC, the most frequent task was using integral properties. Those properties are order of integration, zero width interval, constant multiple, sum, and difference and additivity. Although these properties are important and valuable, students do not have opportunities to explore adding up pieces and the multiplicative relationship.

Use the rules in Table 5.4 and Equations (1)-(3) to evaluate the integrals in Exercises 41-50.

$\int_0^2 5x\,dx$ (TC, p. 271)

Figure 5: Exercise problem from TC

For HC, the most frequent task was evaluating Riemann sum with several rectangles (similar to Figure 4).

Summary and Discussion

With our results, we can think about the types of knowledge that students will be exposed to when they either use these textbooks on their own or their instructors' use these textbooks. With Riemann sum and problems similar to Figure 3, students will be exposed to adding up pieces but it is not clear how students would make a connection between Riemann sum and integral notations. With what we found from these textbooks, they will be familiar with converting Riemann sum to integral, area interpretations, using a few rectangles to find the area under a curve. Although these are important units of knowledge that students need to understand and

they may be able to use and activate those units of knowledge, it is challenging for students to expand their understanding of definite integral as these textbooks provide limited opportunities for them to become familiar with the product layer and adding up pieces. TC is the only textbook that provides learning opportunities to explore both the product layer (definition) and adding up pieces (exercise problems). Although TC's definition shows how $f(c_k)$ are replaced by f(x), how the subinterval widths Δx_k become the differential dx, and what *f(x)* • *dx* means, the exercise problems do not clearly provide students with opportunities to understand the meaning of these symbols. To provide more precise learning opportunities for students, it may be useful to ask them to describe the area of each rectangle, how each area can be expressed as a product of two quantities, and how the product is related to integrand and differential. When these opportunities are added to what three textbooks present to students, they may be able to have units of knowledge needed to understand more clearly what it means to add infinitely many rectangles and how the area of a rectangle can be formed. As described earlier, students struggled most with the product layer but those students who used adding up pieces often correctly identified the product and how it can be interpreted in definite integrals (Jones, 2013; Sealey, 2014). Thus, it will be useful to modify tasks in these textbooks to more clearly make the connection among those rectangles, the product, integrand and differential as researchers recommended (Jones, 2013; Sealey, 2014). Although it is critical to think about how we teach calculus, we can realistically think about what really happens in calculus classrooms. As Wagner (2018) mentioned, when students are able to correctly evaluate definite integral with anti-derivative, they may not feel the need to use and understand the product layer or adding up pieces ideas. Our findings showed one possible area that may contribute to learning challenges in previous studies. Our findings from these widely used textbooks is a first step to think about how to revise and modify curriculum materials to improve teaching and learning of mathematics (Thompson & Harel, 2021) and how our results contribute to further discussion. Thompson and Harel (2021) pointed out that successful curricula require interactions among a triad of elements: a) production of national mathematics curriculum documents, b) mathematics education research, and c) teachers' understandings of content, cognition, and pedagogy. Adding what we know about these textbooks to the results of existing studies is an initial step to think about possible ways to revise integral lessons in calculus textbooks.

References

Bressoud, D. (2011). The calculus I instructor. Retrieved from https://www.maa.org/external_archive/columns/launchings/launchings_06_11.html

Charalambous, C. Y., Delaney, S., Hsu, H.-Y., & Mesa, V. (2010). A Comparative Analysis of the Addition and Subtraction of Fractions in Textbooks from Three Countries. Mathematical Thinking and Learning, 12(2), 117-151. doi:10.1080/10986060903460070

Ely, R. (2017). Definite integral registers using infinitesimals. The Journal of Mathematical Behavior, 48, 152-167. doi:https://doi.org/10.1016/j.jmathb.2017.10.002

Ely, R. (2021). Teaching calculus with infinitesimals and differentials. ZDM – Mathematics Education, 53(3), 591-604. doi:10.1007/s11858-020-01194-2

Harlow, D. B., & Bianchini, J. A. (2020). Knowledge-in-Pieces—Andrea A. diSessa, David Hammer. In B. Akpan & T. J. Kennedy (Eds.), Science Education in Theory and Practice: An Introductory Guide to Learning Theory (pp. 389-401). Cham: Springer International Publishing.

Hong, D. S. (2023). Examining Opportunities to Learn Limit in Widely Used Calculus Textbooks. International Journal of Science and Mathematics Education, 21(3), 881-898. doi:10.1007/s10763-022-10273-7

Hong, D. S., Choi, K. M., Runnalls, C., & Hwang, J. (2018). Do textbooks address known learning challenges in area measurement? A comparative analysis. Mathematics Education Research Journal, 30(3), 325-354. doi:10.1007/s13394-018-0238-6

Jones, S. R. (2013). Understanding the integral: Students' symbolic forms. The Journal of Mathematical Behavior,

32(2), 122-141. doi:https://doi.org/10.1016/j.jmathb.2012.12.004
Jones, S. R. (2015). Areas, anti-derivatives, and adding up pieces: Definite integrals in pure mathematics and applied science contexts. The Journal of Mathematical Behavior, 38, 9-28. doi:https://doi.org/10.1016/j.jmathb.2015.01.001
Larsen, S., Marrongelle, K., Bressoud, D., & Graham, K. (2016). Understanding the Concepts of Calculus: Frameworks and Roadmaps Emerging From Educational Research. In J. Cai (Ed.), Compendium for research in mathematics education (pp. 526-550). Reston, VA: National Council of Teachers of Mathematics.
Mahir, N. (2009). Conceptual and procedural performance of undergraduate students in integration. International Journal of Mathematical Education in Science and Technology, 40(2), 201-211. doi:10.1080/00207390802213591
Rasslan, S., & Tall, D. (2002). Definitions and images for the definite integral concept. Paper presented at the Proceedings of the 26th conference of the International Group for the Psychology of Mathematics Education, Norwich, United Kingdom.
Reeves, C., Carnoy, M., & Addy, N. (2013). Comparing opportunity to learn and student achievement gains in southern African primary schools: A new approach. International Journal of Educational Development, 33(5), 426-435. doi:https://doi.org/10.1016/j.ijedudev.2012.12.006
Remillard, J. T., & Heck, D. J. (2014). Conceptualizing the curriculum enactment process in mathematics education. ZDM, 46(5), 705-718. doi:10.1007/s11858-014-0600-4
Sealey, V. (2014). A framework for characterizing student understanding of Riemann sums and definite integrals. The Journal of Mathematical Behavior, 33, 230-245. doi:https://doi.org/10.1016/j.jmathb.2013.12.002
Sofronas, K. S., DeFranco, T. C., Vinsonhaler, C., Gorgievski, N., Schroeder, L., & Hamelin, C. (2011). What does it mean for a student to understand the first-year calculus? Perspectives of 24 experts. The Journal of Mathematical Behavior, 30(2), 131-148. doi:https://doi.org/10.1016/j.jmathb.2011.02.001
Son, J.-W., & Hu, Q. (2016). The initial treatment of the concept of function in the selected secondary school mathematics textbooks in the US and China. International Journal of Mathematical Education in Science and Technology, 47(4), 505-530. doi:10.1080/0020739X.2015.1088084
Thompson, P. W., & Harel, G. (2021). Ideas foundational to calculus learning and their links to students' difficulties. ZDM – Mathematics Education, 53(3), 507-519. doi:10.1007/s11858-021-01270-1
Wagner, J. F. (2018). Students' Obstacles to Using Riemann Sum Interpretations of the Definite Integral. International Journal of Research in Undergraduate Mathematics Education, 4(3), 327-356. doi:10.1007/s40753-017-0060-7